Analytic geometry, a union of geometry and algebra, enables us to analyze certain geometric concepts algebraically and to interpret certain algebraic relationships geometrically. Our two main concerns center around graphing algebraic equations and finding equations of useful geometric figures. We have discussed a number of topics in analytic geometry, such as straight lines and circles, in earlier chapters. In this chapter we discuss additional analytic geometry topics: conic sections and translation of axes.

René Descartes (1596–1650), the French philosopher–mathematician, is generally recognized as the founder of analytic geometry.

SECTION 11-1 Conic Sections; Parabola

- Conic Sections
- Definition of a Parabola
- Drawing a Parabola
- Standard Equations and Their Graphs
- Applications

In this section we introduce the general concept of a conic section and then discuss the particular conic section called a *parabola*. In the next two sections we will discuss two other conic sections called *ellipses* and *hyperbolas*.

• Conic Sections

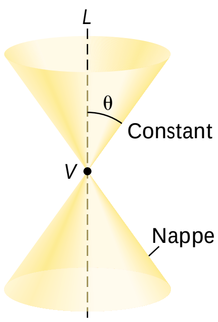
In Section 2-2 we found that the graph of a first-degree equation in two variables,

$$Ax + By = C \tag{1}$$

where A and B are not both 0, is a straight line, and every straight line in a rectangular coordinate system has an equation of this form. What kind of graph will a second-degree equation in two variables,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2}$$

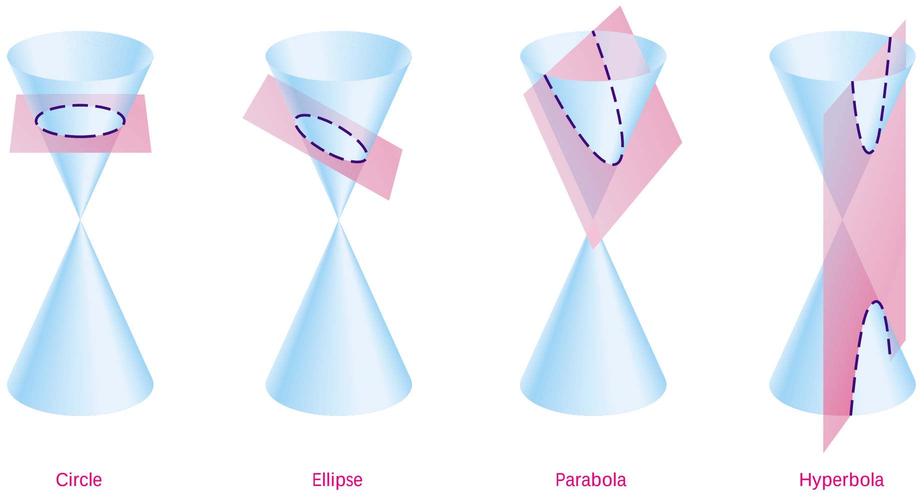
where A, B, and C are not all 0, yield for different sets of values of the coefficients? The graphs of equation (2) for various choices of the coefficients are plane curves obtainable by intersecting a cone* with a plane, as shown in Figure 1. These curves are called **conic sections.**



If a plane cuts clear through one nappe, then the intersection curve is called a **circle** if the plane is perpendicular to the axis and an **ellipse** if the plane is not perpendicular to the axis. If a plane cuts only one nappe, but does not cut clear through,

*Starting with a fixed line L and a fixed point V on L, the surface formed by all straight lines through V making a constant angle θ with L is called a **right circular cone.** The fixed line L is called the axis of the cone, and V is its **vertex.** The two parts of the cone separated by the vertex are called **nappes.**

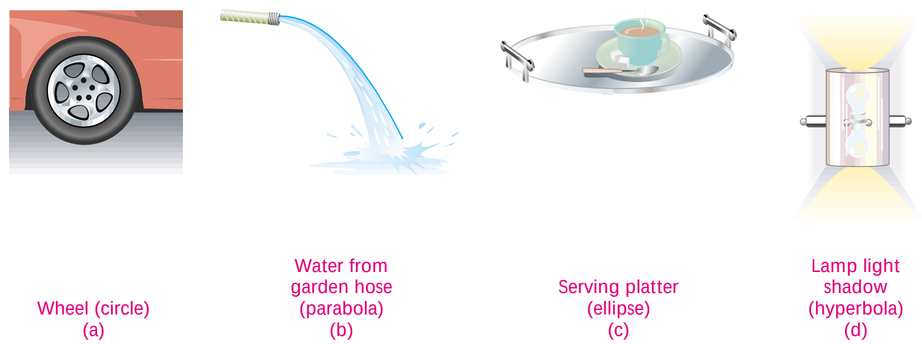
FIGURE 1 Conic sections.



then the intersection curve is called a **parabola.** Finally, if a plane cuts through both nappes, but not through the vertex, the resulting intersection curve is called a **hyperbola.** A plane passing through the vertex of the cone produces a **degenerate conic**—a point, a line, or a pair of lines.

Conic sections are very useful and are readily observed in your immediate surroundings: wheels (circle), the path of water from a garden hose (parabola), some serving platters (ellipses), and the shadow on a wall from a light surrounded by a cylindrical or conical lamp shade (hyperbola) are some examples (see Fig. 2). We will discuss many applications of conics throughout the remainder of this chapter.

FIGURE 2 Examples of conics.



A definition of a conic section that does not depend on the coordinates of points in any coordinate system is called a **coordinate-free definition.** In Section 2-1 we gave a coordinate-free definition of a circle and developed its standard equation in a rectangular coordinate system. In this and the next two sections we will give coordinate-free definitions of a parabola, ellipse, and hyperbola, and we will develop standard equations for each of these conics in a rectangular coordinate system.

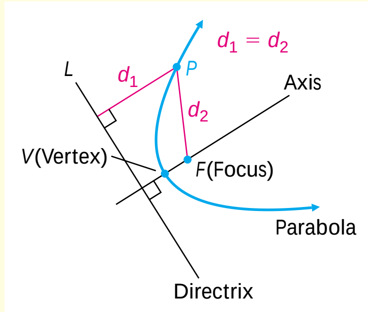
• Definition of a Parabola

The following definition of a parabola does not depend on the coordinates of points in any coordinate system:

DEFINITION 1

Parabola

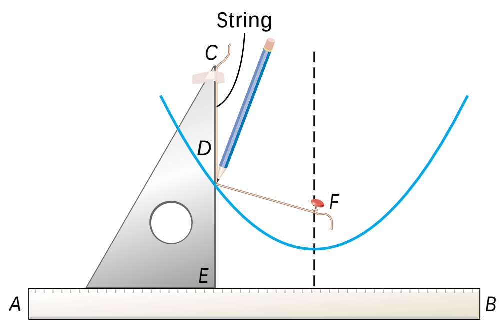
A **parabola** is the set of all points in a plane equidistant from a fixed point F and a fixed line L in the plane. The fixed point F is called the **focus,** and the fixed line L is called the **directrix.** A line through the focus perpendicular to the directrix is called the **axis,** and the point on the axis halfway between the directrix and focus is called the **vertex.**



• Drawing a Parabola

Using the definition, we can draw a parabola with fairly simple equipment—a straightedge, a right-angle drawing triangle, a piece of string, a thumbtack, and a pencil. Referring to Figure 3, tape the straightedge along the line AB and place the thumbtack above the line AB. Place one leg of the triangle along the straightedge as indicated, then take a piece of string the same length as the other leg, tie one end to the thumbtack, and fasten the other end with tape at C on the triangle. Now press the string to the edge of the triangle, and keeping the string taut, slide the triangle along the straightedge. Since DE will always equal DF, the resulting curve will be part of a parabola with directrix AB lying along the straightedge and focus F at the thumbtack.

FIGURE 3 Drawing a parabola.



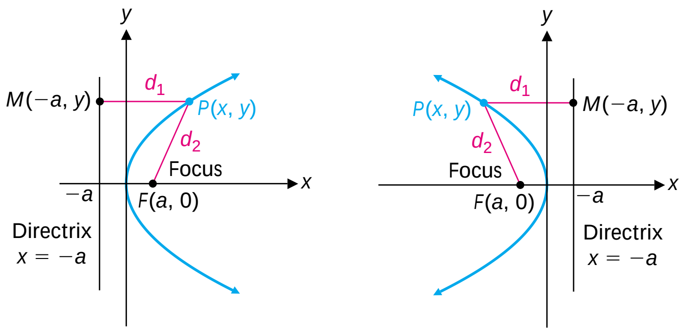
EXPLORE-DISCUSS 1

The line through the focus F that is perpendicular to the axis of a parabola intersects the parabola in two points G and H. Explain why the distance from G to H is twice the distance from F to the directrix of the parabola.

• Standard Equations and Their Graphs

Using the definition of a parabola and the distance-between-two-points formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{3}$$



$a > 0$, focus on positive x axis
(a)

$a < 0$, focus on negative x axis
(b)

FIGURE 4 Parabola with center at the origin and axis the x axis.

we can derive simple standard equations for a parabola located in a rectangular coordinate system with its vertex at the origin and its axis along a coordinate axis. We start with the axis of the parabola along the x axis and the focus at $F(a, 0)$. We locate the parabola in a coordinate system as in Figure 4 and label key lines and points. This is an important step in finding an equation of a geometric figure in a coordinate system. Note that the parabola opens to the right if $a > 0$ and to the left if $a < 0$. The vertex is at the origin, the directrix is $x = -a$, and the coordinates of M are $(-a, y)$.

The point $P(x, y)$ is a point on the parabola if and only if

$$
\begin{aligned}
d_1 &= d_2 \\
d(P, M) &= d(P, F) \\
\sqrt{(x+a)^2 + (y-y)^2} &= \sqrt{(x-a)^2 + (y-0)^2} && \text{Use equation (3).} \\
(x+a)^2 &= (x-a)^2 + y^2 && \text{Square both sides.} \\
x^2 + 2ax + a^2 &= x^2 - 2ax + a^2 + y^2 && \text{Simplify.} \\
y^2 &= 4ax && (4)
\end{aligned}
$$

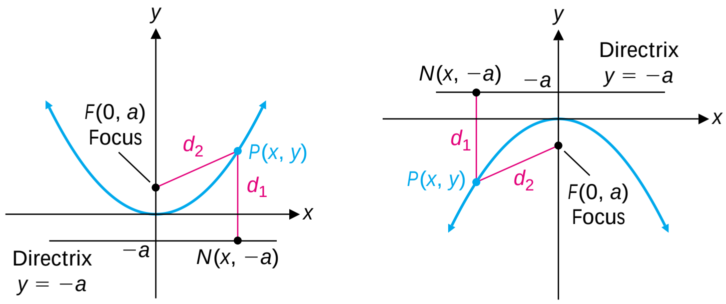
Equation (4) is the standard equation of a parabola with vertex at the origin, axis the x axis, and focus at $(a, 0)$.

Now we locate the vertex at the origin and focus on the y axis at $(0, a)$. Looking at Figure 5 on the following page, we note that the parabola opens upward if $a > 0$ and downward if $a < 0$. The directrix is $y = -a$, and the coordinates of N are $(x, -a)$. The point $P(x, y)$ is a point on the parabola if and only if

$$
\begin{aligned}
d_1 &= d_2 \\
d(P, N) &= d(P, F) \\
\sqrt{(x-x)^2 + (y+a)^2} &= \sqrt{(x-0)^2 + (y-a)^2} && \text{Use equation (3).} \\
(y+a)^2 &= x^2 + (y-a)^2 && \text{Square both sides.} \\
y^2 + 2ay + a^2 &= x^2 + y^2 - 2ay + a^2 && \text{Simplify.} \\
x^2 &= 4ay && (5)
\end{aligned}
$$

Equation (5) is the standard equation of a parabola with vertex at the origin, axis the y axis, and focus at $(0, a)$.

FIGURE 5 Parabola with center at the origin and axis the y axis.



$a > 0$, focus on positive y axis
(a)

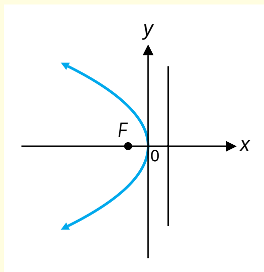
$a < 0$, focus on negative y axis
(b)

We summarize these results for easy reference in Theorem 1:

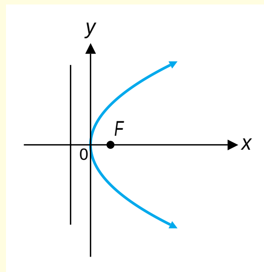
Theorem 1

Standard Equations of a Parabola with Vertex at (0, 0)

1. $y^2 = 4ax$
Vertex: (0, 0)
Focus: $(a, 0)$
Directrix: $x = -a$
Symmetric with respect to the x axis
Axis the x axis

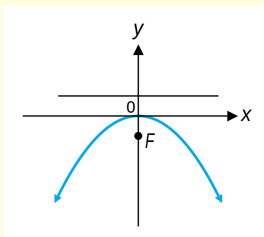


$a < 0$ (opens left)

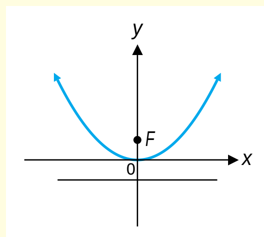


$a > 0$ (opens right)

2. $x^2 = 4ay$
Vertex: (0, 0)
Focus: $(0, a)$
Directrix: $y = -a$
Symmetric with respect to the y axis
Axis the y axis



$a < 0$ (opens down)



$a > 0$ (opens up)

EXAMPLE 1 Graphing $x^2 = 4ay$

Graph $x^2 = -16y$, and locate the focus and directrix.

Solution To graph $x^2 = -16y$, it is convenient to assign y values that make the right side a perfect square, and solve for x. Note that y must be 0 or negative for x to be real. Since the coefficient of y is negative, a must be negative, and the parabola opens downward (Fig. 6).

x	0	± 4	± 8
y	0	-1	-4

Focus: $x^2 = -16y = 4(\overset{a}{-4})y$

$F(0, a) = F(0, -4)$

Directrix: $y = -a$

$= -(-4) = 4$

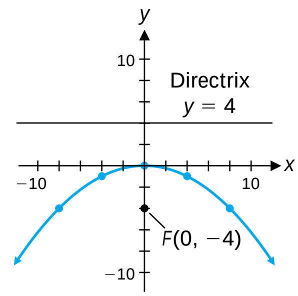
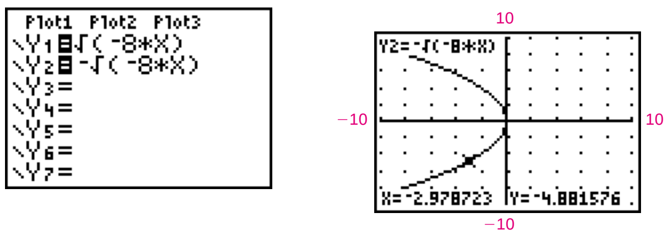


FIGURE 6 $x^2 = -16y$.

Matched Problem 1 Graph $y^2 = -8x$, and locate the focus and directrix.

Remark. To graph the equation $x^2 = -16y$ of Example 1 on a graphing utility, we first solve the equation for y and then graph the function $y = -\frac{1}{16}x^2$. If that same approach is used to graph the equation $y^2 = -8x$ of Matched Problem 1, then $y = \pm\sqrt{-8x}$, and there are two functions to graph. The graph of $y = \sqrt{-8x}$ is the upper half of the parabola, and the graph of $y = -\sqrt{-8x}$ is the lower half (see Fig. 7).

FIGURE 7



CAUTION A common error in making a quick sketch of $y^2 = 4ax$ or $x^2 = 4ay$ is to sketch the first with the y axis as its axis and the second with the x axis as its axis. The graph of $y^2 = 4ax$ is symmetric with respect to the x axis, and the graph of $x^2 = 4ay$ is symmetric with respect to the y axis, as a quick symmetry check will reveal.

EXAMPLE 2 Finding the Equation of a Parabola

(A) Find the equation of a parabola having the origin as its vertex, the y axis as its axis, and $(-10, -5)$ on its graph.

(B) Find the coordinates of its focus and the equation of its directrix.

Solutions (A) The parabola is opening down and has an equation of the form $x^2 = 4ay$. Since $(-10, -5)$ is on the graph, we have

$$\begin{aligned} x^2 &= 4ay \\ (-10)^2 &= 4a(-5) \\ 100 &= -20a \\ a &= -5 \end{aligned}$$

Thus, the equation of the parabola is

$$\begin{aligned} x^2 &= 4(-5)y \\ &= -20y \end{aligned}$$

(B) Focus: $x^2 = -20y \qquad = 4(\overset{a}{-5})y$

$F(0, a) = F(0, -5)$

Directrix: $\begin{aligned} y &= -a \\ &= -(-5) \\ &= 5 \end{aligned}$

Matched Problem 2 (A) Find the equation of a parabola having the origin as its vertex, the x axis as its axis, and $(4, -8)$ on its graph.

(B) Find the coordinates of its focus and the equation of its directrix.

EXPLORE-DISCUSS 2 Consider the graph of an equation in the variables x and y. The equation of its magnification by a factor $k > 0$ is obtained by replacing x and y in the equation by x/k and y/k, respectively. (Of course, a magnification by a factor k between 0 and 1 means an actual reduction in size.)

(A) Show that the magnification by a factor 3 of the circle with equation $x^2 + y^2 = 1$ has equation $x^2 + y^2 = 9$.

(B) Explain why every circle with center at (0, 0) is a magnification of the circle with equation $x^2 + y^2 = 1$.

(C) Find the equation of the magnification by a factor 3 of the parabola with equation $x^2 = y$. Graph both equations.

(D) Explain why every parabola with vertex (0, 0) that opens upward is a magnification of the parabola with equation $x^2 = y$.

• Applications

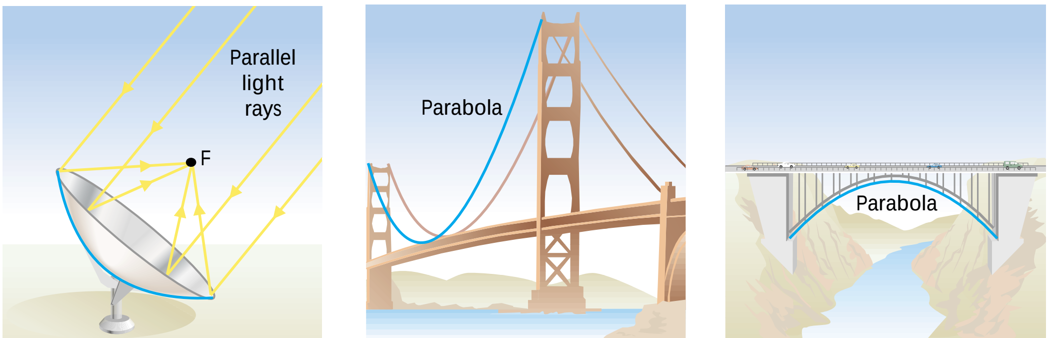
Parabolic forms are frequently encountered in the physical world. Suspension bridges, arch bridges, microphones, symphony shells, satellite antennas, radio and optical telescopes, radar equipment, solar furnaces, and searchlights are only a few of many items that utilize parabolic forms in their design.

Figure 8(a) illustrates a parabolic reflector used in all reflecting telescopes—from 3- to 6-inch home type to the 200-inch research instrument on Mount Palomar in California. Parallel light rays from distant celestial bodies are reflected to the focus off a parabolic mirror. If the light source is the sun, then the parallel rays are focused at F and we have a solar furnace. Temperatures of over 6,000°C have been achieved by such furnaces. If we locate a light source at F, then the rays in Figure 8(a) reverse, and we have a spotlight or a searchlight. Automobile headlights can use parabolic reflectors with special lenses over the light to diffuse the rays into useful patterns.

Figure 8(b) shows a suspension bridge, such as the Golden Gate Bridge in San Francisco. The suspension cable is a parabola. It is interesting to note that a free-hanging cable, such as a telephone line, does not form a parabola. It forms another curve called a *catenary.*

Figure 8(c) shows a concrete arch bridge. If all the loads on the arch are to be compression loads (concrete works very well under compression), then using physics and advanced mathematics, it can be shown that the arch must be parabolic.

FIGURE 8 Uses of parabolic forms.



Parabolic reflector (a) | Suspension bridge (b) | Arch bridge (c)

EXAMPLE 3 Parabolic Reflector

A **paraboloid** is formed by revolving a parabola about its axis. A spotlight in the form of a paraboloid 5 inches deep has its focus 2 inches from the vertex. Find, to one decimal place, the radius R of the opening of the spotlight.

Solution

Step 1. Locate a parabolic cross section containing the axis in a rectangular coordinate system, and label all known parts and parts to be found. This is a very important step and can be done in infinitely many ways. Since we are in charge, we can make things simpler for ourselves by locating the vertex at the origin and choosing a coordinate axis as the axis. We choose the y axis as the axis of the parabola with the parabola opening upward. See Figure 9 on the following page.

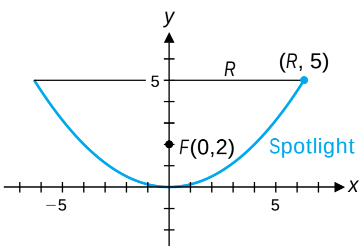


FIGURE 9

Step 2. Find the equation of the parabola in the figure. Since the parabola has the y axis as its axis and the vertex at the origin, the equation is of the form

$$x^2 = 4ay$$

We are given $F(0, a) = F(0, 2)$; thus, $a = 2$, and the equation of the parabola is

$$x^2 = 8y$$

Step 3. Use the equation found in step 2 to find the radius R of the opening. Since $(R, 5)$ is on the parabola, we have

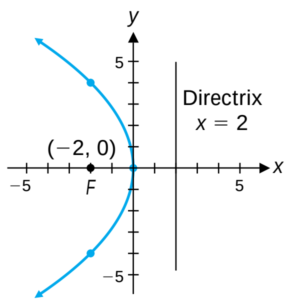
$$R^2 = 8(5)$$
$$R = \sqrt{40} \approx 6.3 \text{ inches}$$

Matched Problem 3 Repeat Example 3 with a paraboloid 12 inches deep and a focus 9 inches from the vertex.

Answers to Matched Problems

1. Focus: $(-2, 0)$
Directrix: $x = 2$

x	0	-2
y	0	± 4



2. (A) $y^2 = 16x$ (B) Focus: $(4, 0)$; Directrix: $x = -4$
3. $R = 20.8$ in.

EXERCISE 11-1

A

In Problems 1–12, graph each equation, and locate the focus and directrix.

1. $y^2 = 4x$ **2.** $y^2 = 8x$ **3.** $x^2 = 8y$
4. $x^2 = 4y$ **5.** $y^2 = -12x$ **6.** $y^2 = -4x$
7. $x^2 = -4y$ **8.** $x^2 = -8y$ **9.** $y^2 = -20x$
10. $x^2 = -24y$ **11.** $x^2 = 10y$ **12.** $y^2 = 6x$

Find the coordinates to two decimal places of the focus for each parabola in Problems 13–18.

13. $y^2 = 39x$ **14.** $x^2 = 58y$ **15.** $x^2 = -105y$
16. $y^2 = -93x$ **17.** $y^2 = -77x$ **18.** $x^2 = -205y$

B

In Problems 19–24, find the equation of a parabola with vertex at the origin, axis the x or y axis, and:

19. Focus $(-6, 0)$

20. Directrix $y = 8$

21. Directrix $y = -5$

22. Focus $(3, 0)$

23. Focus $(0, -\frac{1}{3})$

24. Directrix $y = -\frac{1}{2}$

In Problems 25–30, find the equation of the parabola having its vertex at the origin, its axis as indicated, and passing through the indicated point.

25. x axis; $(-4, -20)$

26. y axis; $(30, -15)$

27. y axis; $(9, -27)$

28. x axis; $(121, 11)$

29. x axis; $(-8, -2)$

30. y axis; $(-\sqrt{2}, 3)$

In Problems 31–34, find the first-quadrant points of intersection for each system of equations to three decimal places.

Check Problems 31–34 with a graphing utility.

31. $x^2 = 4y$
$y^2 = 4x$

32. $y^2 = 3x$
$x^2 = 3y$

33. $y^2 = 6x$
$x^2 = 5y$

34. $x^2 = 7y$
$y^2 = 2x$

35. Consider the parabola with equation $x^2 = 4ay$.
(A) How many lines through $(0, 0)$ intersect the parabola in exactly one point? Find their equations.
(B) Find the coordinates of all points of intersection of the parabola with the line through $(0, 0)$ having slope $m \neq 0$.

36. Find the coordinates of all points of intersection of the parabola with equation $x^2 = 4ay$ and the parabola with equation $y^2 = 4bx$.

37. If a line through the focus contains two points A and B of a parabola, then the line segment AB is called a **focal chord**. Find the coordinates of A and B for the focal chord that is perpendicular to the axis of the parabola $x^2 = 4ay$.

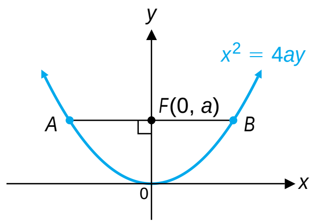


Figure for 37 and 38

38. Find the length of the focal chord AB that is perpendicular to the axis of the parabola $x^2 = 4ay$.

In Problems 39–42, determine whether the statement is true or false. If true, explain why. If false, give a counterexample.

39. If a is real, then the graph of $y^2 = 4ax$ is a parabola.

40. If a is negative, then the graph of $y^2 = 4ax$ is a parabola.

41. Every vertical line intersects the graph of $x^2 = 4y$.

42. Every nonhorizontal line intersects the graph of $x^2 = 4y$.

C

In Problems 43–46, use the definition of a parabola and the distance formula to find the equation of a parabola with:

43. Directrix $y = -4$ and focus $(2, 2)$

44. Directrix $y = 2$ and focus $(-3, 6)$

45. Directrix $x = 2$ and focus $(6, -4)$

46. Directrix $x = -3$ and focus $(1, 4)$

In Problems 47–50, use a graphing utility to find the coordinates of all points of intersection to two decimal places.

47. $x^2 = 8y, y = 5x + 4$

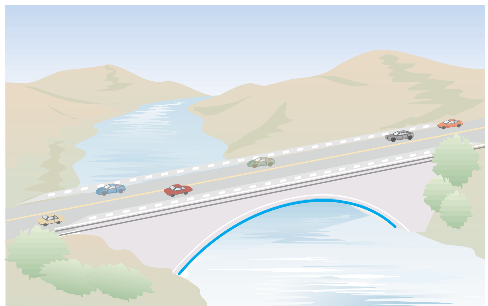
48. $x^2 = 3y, 7x + 4y = 11$

49. $x^2 = -8y, y^2 = -5x$

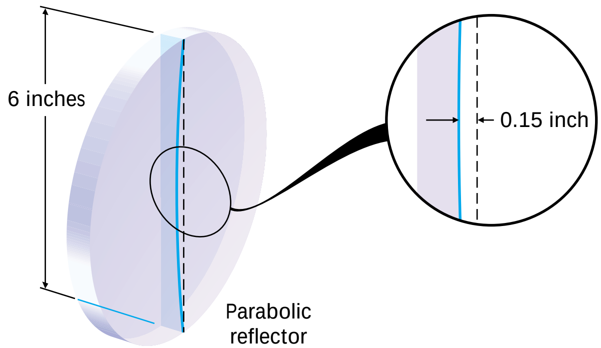
50. $y^2 = 6x, 2x - 9y = 13$

APPLICATIONS

51. Engineering. The parabolic arch in the concrete bridge in the figure must have a clearance of 50 feet above the water and span a distance of 200 feet. Find the equation of the parabola after inserting a coordinate system with the origin at the vertex of the parabola and the vertical y axis (pointing upward) along the axis of the parabola.

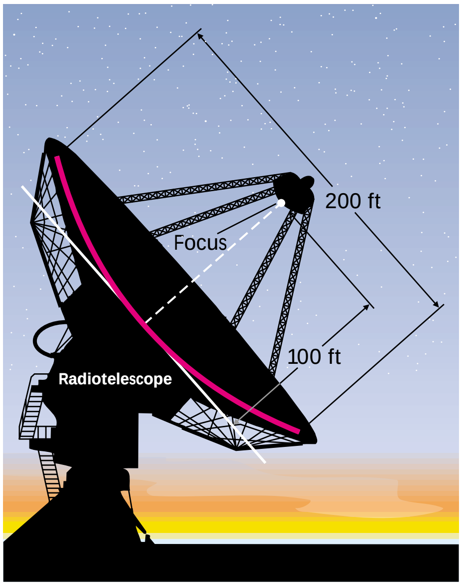


52. Astronomy. The cross section of a parabolic reflector with 6-inch diameter is ground so that its vertex is 0.15 inch below the rim (see the figure).



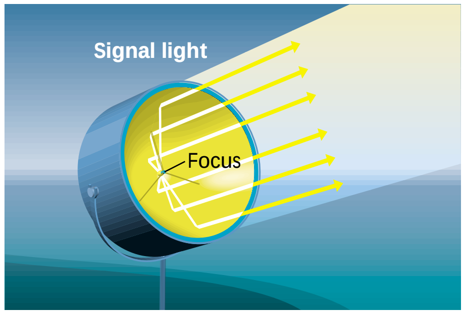
(A) Find the equation of the parabola after inserting an xy coordinate system with the vertex at the origin, the y axis (pointing upward) the axis of the parabola.
(B) How far is the focus from the vertex?

53. Space Science. A designer of a 200-foot-diameter parabolic electromagnetic antenna for tracking space probes wants to place the focus 100 feet above the vertex (see the figure).



(A) Find the equation of the parabola using the axis of the parabola as the y axis (up positive) and vertex at the origin.
(B) Determine the depth of the parabolic reflector.

54. Signal Light. A signal light on a ship is a spotlight with parallel reflected light rays (see the figure). Suppose the parabolic reflector is 12 inches in diameter and the light source is located at the focus, which is 1.5 inches from the vertex.



(A) Find the equation of the parabola using the axis of the parabola as the x axis (right positive) and vertex at the origin.
(B) Determine the depth of the parabolic reflector.

SECTION 11-2 Ellipse

- Definition of an Ellipse
- Drawing an Ellipse
- Standard Equations and Their Graphs
- Applications

We start our discussion of the ellipse with a coordinate-free definition. Using this definition, we show how an ellipse can be drawn and we derive standard equations for ellipses specially located in a rectangular coordinate system.

• Definition of an Ellipse

The following is a coordinate-free definition of an ellipse:

DEFINITION 1

Ellipse

An **ellipse** is the set of all points P in a plane such that the sum of the distances of P from two fixed points in the plane is constant. Each of the fixed points, F' and F, is called a **focus,** and together they are called **foci.** Referring to the figure, the line segment $V'V$ through the foci is the **major axis.** The perpendicular bisector $B'B$ of the major axis is the **minor axis.** Each end of the major axis,